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Eigenvectors of Backwardshift on a Deformed Hilbert Space

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We study reproducing kernel and Coherent vectors in a deformed Hilbert space.

We consider the set

$$H_q = \{f: f(z) = \sum a_n z^n \text{ where } \Sigma[n]! \left|a_n\right|^2 < \infty\}$$

where $[n] = (1 - q^n)/(1 - q)$, $0 < q < 1$.

For $f, g \in H_q$, $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$, $g(z) = \Sigma_{n=0}^{\infty} b_n z^n$ we define addition and scalar multiplication as follows:

$$f(z) + g(z) = \sum_{n=0}^{\infty} (a_n + b_n) z^n \tag{1}$$

and

$$\lambda \circ f(z) = \sum_{n=0}^{\infty} \lambda a_n \tag{2}$$

It is easily seen that H_q forms a vector space with respect to usual pointwise scalar multiplication and pointwise addition by (1) and (2). We observe that $e_q(z) = \Sigma_{n=0}^{\infty} (z^n/[n]!)$.

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Now we define the inner product of two functions $f(z) = \Sigma\, a_n z^n$ and $g(z) = \Sigma\, b_n z^n$ belonging to H_q as

$$(f, g) = \sum [n]!\, \bar{a}_n b_n \tag{3}$$

The corresponding norm is given by

$$\|f\|^2 = (f, f) = \sum [n]!\, |a_n|^2 < \infty$$

With this norm derived from the inner product it can be shown that H_q is a complete normed space. Hence H_q forms a Hilbert space.

Proposition 1. The set $\{z^n/\sqrt{[n]!},\ n = 1, 2, 3, \ldots\}$ forms a complete orthonormal set.

Proof. If $f_n = z^n/\sqrt{[n]!}$, $n = 1, 2, 3, \ldots$, then

$$\|f_n\| = (f_n, f_n)^{1/2} = 1$$

and $(f_n, f_m) = 0$. Hence $\{f_n\}$ forms an orthonormal set. Also, it is complete, for if $f(z) = \Sigma\, a_n z^n \in H_q$, then

$$(f_n, f) = [n]!\, a_n \frac{1}{\sqrt{[n]!}} = \sqrt{[n]!}\, a_n$$

Hence

$$\Sigma\, |(f_n, f)|^2 = \sum [n]!\, |a_n|^2 = \|f\|^2$$

By Parseval's theorem, $\{f_n\}$ is complete.

1. REPRODUCING KERNEL

Since H_q is a functional Hilbert space, the linear functional $f \to f(z)$ on H_q is bounded for every $z \in \mathbb{C}$. Consequently, there exists, for each $z \in \mathbb{C}$, an element K_z of H_q such that $f(z) = (K_z, f)$ for all $f \in H_q$. The function $K(w, z) = K_z(w)$ is called the kernel function or the reproducing kernel of H_q.

Consider the Fourier expansion of K_z with respect to the orthonormal basis $f_n(z)$:

$$K_z = \sum_n (f_n, K_z) f_n = \sum \bar{f}_n(z) f_n$$

Hence

$$K(w, z) = K_z(w) = (K_w, K_z) = \sum f_n(w) \bar{f}_n(z)$$

$$= \sum \frac{w^n}{\sqrt{[n]!}} \frac{\bar{z}^n}{\sqrt{[n]!}} = \sum \frac{(\bar{z}w)^n}{[n]!} = e_q(\bar{z}w)$$

Thus $K(w, z) = e_q(\bar{z}w)$ is the reproducing kernel for H_q.

2. EIGENVECTORS

We consider the following actions on H_q:

$$\begin{aligned} Tf_n &= \sqrt{[n]}\, f_{n-1} \\ T^*f_n &= \sqrt{[n+1]}\, f_{n+1} \end{aligned} \tag{4}$$

T is the backward shift and its adjoint T^* is the forward shift operator on H_q.

2.1. Backwardshift

Now we shall find the solution of the following eigenvalue equation:

$$Tf = \alpha f \tag{5}$$

We have

$$f(\alpha) = \sum_{n=0}^{\infty} a_n \alpha^n = \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n \tag{6}$$

$$\begin{aligned} Tf(\alpha) &= \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, Tf_n \\ &= \sum_{n=1}^{\infty} a_n \sqrt{[n]!}\, \sqrt{[n]}\, f_{n-1} \\ &= \sum_{n=0}^{\infty} a_{n+1} \sqrt{[n+1]!}\, \sqrt{[n+1]}\, f_n \end{aligned} \tag{7}$$

$$\alpha f(\alpha) = \alpha \sum_{n=0}^{\infty} a_n \alpha^n = \alpha \sum_{n=0}^{\infty} a_n \sqrt{[n]!}\, f_n \tag{8}$$

From (5)–(8) we observe that a_n satisfies the following difference equation:

$$a_{n+1} \sqrt{[n+1]}\, \sqrt{[n+1]} = \alpha a_n \tag{9}$$

That is,

$$a_{n+1} = \frac{\alpha a_n}{[n+1]} \tag{10}$$

Hence,

$$a_1 = \frac{\alpha a_0}{[1]}, \qquad a_2 = \frac{\alpha a_1}{[2]} = \frac{\alpha^2 a_0}{[2]!}, \qquad a_3 = \frac{\alpha a_2}{[3]} = \frac{\alpha^3 a_0}{[3]!}, \ldots$$

Thus,

$$a_n = \frac{\alpha^n a_0}{[n]!}$$

Hence,

$$f(\alpha) = \sum a_n \sqrt{[n]!}\, f_n = a_0 \sum \frac{\alpha^n}{\sqrt{[n]!}} f_n$$

We choose a_0 so that $f(\alpha)$ is normalized:

$$1 = (f(\alpha), f(\alpha)) = \sum [n]!\, |a_n|^2 = \sum [n]! \frac{|a_0|^2 |\alpha|^{2n}}{([n]!)^2}$$

$$= |a_0|^2 \sum \frac{(|\alpha|^2)^n}{[n]!} = |a_0|^2 e_q(|\alpha|^2)$$

Thus, aside from a trivial phase

$$a_n = e_q\,(|\alpha|^2)^{-1/2} \frac{\alpha^n}{[n]!}$$

So, the eigenvector of T is

$$f(\alpha) = e_q\,(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{[n]!}} f_n \tag{11}$$

We shall call $f(\alpha)$ a **coherent vector** in H_q.

2.2. Square of Backwardshift

Here we shall find the following eigenvalue equation:

$$T^2 f = \alpha^2 f \tag{12}$$

We have

$$\begin{aligned} T^2 f(\alpha) &= T \sum_{n=0}^{\infty} a_{n+1} \sqrt{[n+1]!}\, \sqrt{[n+1]}\, f_n \\ &= \sum_{n=1}^{\infty} a_{n+1} \sqrt{[n+1]!}\, \sqrt{[n+1[}\, \sqrt{[n]}\, f_{n-1} \qquad (13) \\ &= \sum_{n=0}^{\infty} a_{n+2} \sqrt{[n+2]!}\, \sqrt{[n+2]}\, \sqrt{[n+1]}\, f_n \end{aligned}$$

$$\alpha^2 f(\alpha) = \sum_{n=0}^{\infty} \alpha^2 a_n \sqrt{[n]!}\, f_n \tag{14}$$

From (12)–(14) we see that a_n satisfies the following difference equation:

$$a_{n+2}\sqrt{[n+2!}\ \sqrt{[n+2]}\ \sqrt{[n+1]} = \alpha^2 a_n \sqrt{[n]!}$$

Thus,

$$a_{n+2} = \frac{\alpha^2 a_n}{[n+2][n+1]} \tag{15}$$

Hence,

$$a_2 = \frac{\alpha^2 a_0}{[2]!}, \qquad a_4 = \frac{\alpha^2 a_2}{[4][3]} = \frac{\alpha^4 a_0}{[4]!}, \qquad a_6 = \frac{\alpha^2 a_4}{[6][5]} = \frac{\alpha^6 a_0}{[6]!}, \ldots$$

and

$$a_3 = \frac{\alpha^2 a_1}{[3][2]} = \frac{\alpha^2 a_1}{[3]!}, \qquad a_5 = \frac{\alpha^2 a_3}{[5][4]} = \frac{\alpha^4 a_1}{[5]!}, \ldots$$

Thus,

$$\begin{aligned}
f(\alpha) &= a_0 f_0 + a_1 f_1 + a_2\sqrt{[2]!}\, f_2 + a_3\sqrt{[3]!}\, f_3 + \cdots \\
&= (a_0 f_0 + a_2\sqrt{[2]!}\, f_2 + a_4\sqrt{[4]!}\, f_4 + \cdots) \\
&\quad + (a_1 f_1 + a_3\sqrt{[3]!}\, f_3 + a_5\sqrt{[5]!}\, f_5 + \cdots) \\
&= a_0\left(f_0 + \frac{\alpha^2}{\sqrt{[2]!}} f_2 + \frac{\alpha^4}{\sqrt{[4]!}} f_4 + \cdots\right) \\
&\quad + a_1\left(f_1 + \frac{\alpha^2}{\sqrt{[3]!}} f_3 + \frac{\alpha^4}{\sqrt{[5]!}} f_5 + \cdots\right) \\
&= a_0\left(\frac{g(\alpha) + g(-\alpha)}{2N}\right) + \frac{a_1}{\alpha}\left(\frac{g(\alpha) - g(-\alpha)}{2N}\right) \\
&= \left(\frac{a_0}{2N} + \frac{a_1}{2\alpha N}\right) g(\alpha) + \left(\frac{a_0}{2N} - \frac{a_1}{2\alpha N}\right) g(-\alpha) \\
&= K g(\alpha) + K' g(-\alpha)
\end{aligned}$$

where we have taken $g(\alpha) = e_q[|\alpha|^2]$. $f_1(\alpha)$ and $f_1(\alpha)$ satisfy equation (5). Also we have taken $K = a_0/2N + a_1/(2\alpha N)$ and $K' = a_0/2N - a_1/(2\alpha N)$ with $N = e_q(|\alpha|^2)^{-1/2}$.

We choose a_0 and a_1 so that $f(\alpha)$ is normalized:

$$1 = (f(\alpha), f(\alpha)) = e_q^{|\alpha|^2}[|K|^2 + |K'|^2] + e_q^{-|\alpha|^2}[2 \operatorname{Re} K\overline{K}']$$

where we have used the facts

$$(g(\alpha), g(\alpha)) = e_q^{|\alpha|^2}$$
$$(g(-\alpha), g(-\alpha)) = e_q^{|\alpha|^2}$$
$$(g(\alpha), g(-\alpha)) = e_q^{-|\alpha|^2}$$

3. PROPERTIES OF COHERENT VECTORS

Coherent vectors are not orthogonal, for

$$(f(\alpha), f(\alpha')) = e_q(|\alpha|^2)^{-1/2} \cdot e_q(|\alpha'|^2)^{-1/2} \sum_{n=0}^{\infty} [n]! \frac{\overline{\alpha}^n}{[n]!} \frac{\alpha'^n}{[n]!}$$
$$= e_q(|\alpha|^2)^{-1/2} e_q(|\alpha'|^2) \cdot e_q(\overline{\alpha}\alpha') \tag{16}$$

Nevertheless, the coherent vectors are complete, in fact, overcomplete—they form a resolution of the identity

$$I = \frac{1}{\pi} \int_{\alpha \in \mathbb{C}} d\mu(\alpha) \, |f(\alpha)\rangle\langle f(\alpha)| \tag{17}$$

where $d\mu(\alpha) = e_q(|\alpha|^2) \; e_q(-|\alpha|^2) \; \gamma \; d_q|\alpha| d\theta$ with $\alpha = \gamma e^{2\theta}$.

To prove this we define the operator

$$|f(\alpha)\rangle\langle f(\alpha)|: \quad H_q \to H_q \tag{18}$$

by

$$|f(\alpha)\rangle\langle f(\alpha)| f(z) = (f(\alpha), f(z)) f(\alpha) \tag{19}$$

with $f(z) = \sum_{j}^{\infty} b_n x^n$.

Now,

$$(f(\alpha), f(z)) = e_q(|\alpha|^2)^{-1/2} \sum_{n=0}^{\infty} [n]! \frac{\overline{\alpha}^n}{[n]!} b_n$$

Then,

$$(f(\alpha), f(z)) f(\alpha) = e_q(|\alpha|^2)^{-1} \sum_{m,n=0}^{\infty} \frac{\alpha^m}{\sqrt{[m]!}} \overline{\alpha}^n b_n f_m$$

Hence,

$$\frac{1}{\pi}\int_{\alpha\in\mathbb{C}} d\mu\, \big|f(\alpha)\rangle\langle f(\alpha)\big|f(z) = \sum_{m,n=0}^{\infty} \frac{f_m}{\sqrt{[m]!}}\, b_n\, \frac{1}{\pi}\int_0^{\infty} r\, d_q r\, e_q^{-r^2} r^{m+n} \times \int_0^{2\pi} d\theta\, e^{i(m-n)\theta}$$

$$= \sum_{n=0}^{\infty} \frac{f_n}{\sqrt{[n]!}}\, b_n \int_0^{\infty} 2r\, d_q r\, e_q^{-r^2} r^{2n}$$

$$= \sum_{n=0}^{\infty} \frac{f_n}{\sqrt{[n]!}}\, b_n \int_0^{\infty} d_q x\, e_q^{-x} x^n$$

$$= \sum_{n=0}^{\infty} \sqrt{[n]!}\, b_n f_n$$

$$= f(z) \tag{20}$$

where we have taken $x = r^2$ and utilized the fact that $\int_0^\infty d_q x\, e_q^{-x} x^n = [n]!$ (Gray and Nelson, 1990).

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